Math 429 - Exercise Sheet 7

1. Show that if $W \subseteq V$ are representations of a Lie algebra $\mathfrak{g}$, then the associated s.i.b.f.'s satisfy

$$(x, y)_V = (x, y)_W + (x, y)_{V/W}$$

2. Show that if $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{i}^\perp$ (with respect to any s.i.b.f.) is also an ideal.

3. Show that if $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then

$$(x, y)_{\mathfrak{i}} = (x, y)_{\mathfrak{g}}$$

for all $x, y \in \mathfrak{i} \subseteq \mathfrak{g}$.

4. Prove that $\mathfrak{sl}_n$ is a simple Lie algebra (*Hint: take any non-zero $X \in \mathfrak{sl}_n$, and show that you can obtain any E_{ij}, $i \neq j$ from X by suitably taking commutators*).

5. Because of the previous problem, Lemma 2 implies that the Killing form of $\mathfrak{sl}_n$ must be equal to a constant times the s.i.b.f. $(X, Y) \mapsto \mathrm{tr}(XY)$. Calculate the constant in question.

6. For $\mathfrak{g} \in \{\mathfrak{o}_n, \mathfrak{sp}_{2n}\}$, check that the s.i.b.f. $(X, Y) \mapsto \mathrm{tr}(XY)$ is non-degenerate. Here we are considering the trace on $n \times n$ matrices (in the case $\mathfrak{o}_n$) and on $2n \times 2n$ matrices (in the case $\mathfrak{sp}_{2n}$).

7. If X is a strictly upper triangular $n \times n$ matrix, prove that

$$\mathrm{ad}_X : \mathfrak{gl}_n \to \mathfrak{gl}_n, \qquad \mathrm{ad}_X(Y) = [X, Y]$$

is a nilpotent operator (thus establishing the first blue claim in Lecture 7).

(*) Prove the first blue claim in Subsection 7.3 of the lecture notes: "*However, you can prove by analogy with Theorem 11 that any $y \in \mathfrak{g} \backslash rad(\mathfrak{g})$ also sends W to W*".